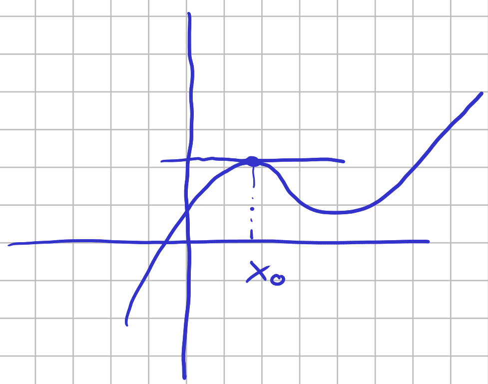
S2.3 Extremwerte, Grenzwerte und Kurvendiskussion

Extremwerte

Sei $f : D_f \to \mathbb{R}$. Ein Wert $x_0 \in D_f$ mit $f'(x_0) = 0$ heißt Extremwert von f



Satz

x_0 Extremwert von f mit

$f''(x) < 0$ ist lokales Max von f

$f''(x) > 0$ " " Min " "

Bsp: $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$

$f'(x) \overset{!}{=} 0 \iff 3x^2 - 3 = 0 \iff x^2 = 1$

$\iff x = \pm 1$

zwei Extremwerte

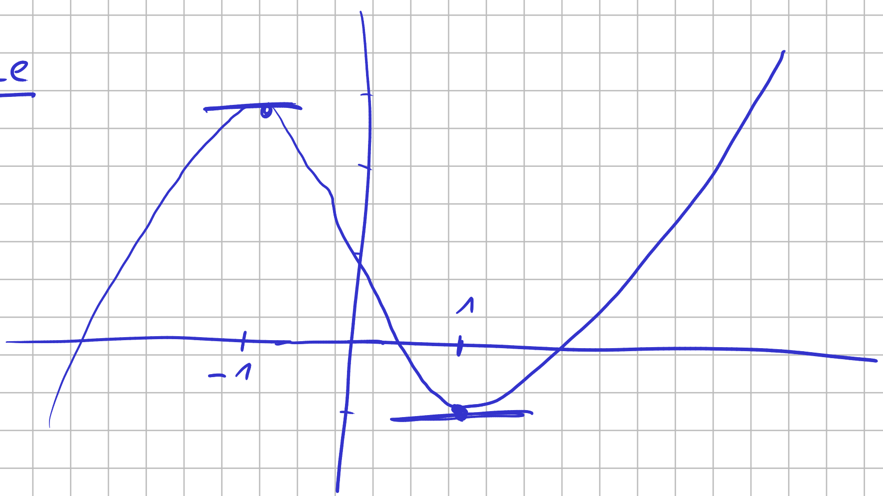
$f''(x) = 6x \Rightarrow f''(-1) = -6 < 0$

$f''(1) = 6 > 0$

Also lokale Maximum bei $(-1, f(-1)) = (-1, (-1)^3 - 3(-1) + 1)$
$= (-1, -1 + 3 + 1)$
$= (-1, 3)$

" " Minimum bei $(1, f(1)) = (1, 1^3 - 3 + 1)$
$= (1, -1)$

<u>Skizze</u>



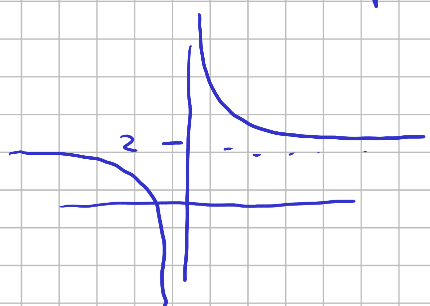
<u>Grenzwerte:</u> $f : D_f \to \mathbb{R}$

im unendlichen $\left([a, \infty) \subseteq D_f \text{ bzw. } (-\infty, a] \subseteq D_f \right)$

$\lim\limits_{x \to \infty} f(x) = a$ f „geht" nach a für $x \to \infty$

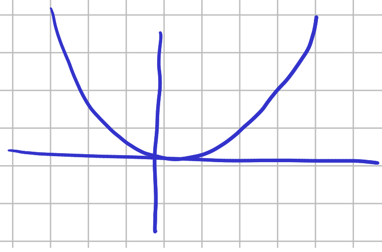
$\lim\limits_{x \to -\infty} f(x) = a$. f " " " für $x \to -\infty$

<u>Bsp:</u> $\lim\limits_{x \to \infty} \frac{1}{x} + 2 = 2$



$\lim\limits_{x \to -\infty} \frac{1}{x} + 2 = 2$

$\lim\limits_{x \to \pm\infty} x^2 = \infty$



Im endlichen

$\lim_{x \to b^+} f(x) = a$ f „geht" gegen a für $x \to b$ von rechts

Rechtsseitiger Grenzwert

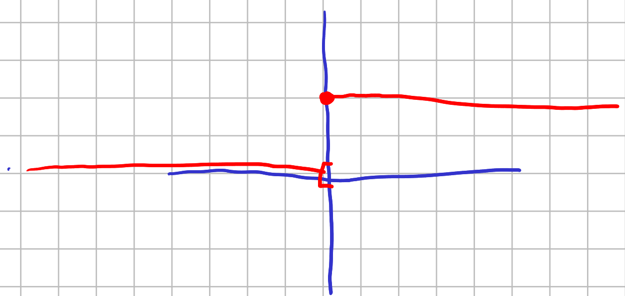
$\lim_{x \to b^-} f(x) = a$ f " " von links

Linksseitiger Grenzwert

Beispiel:

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

Heaviside Fkt

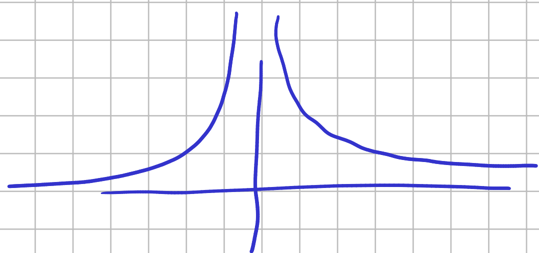


$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = 0$

Falls $\underbrace{\lim_{x \to b^-} f(x) = \lim_{x \to b^+} f(x)}_{=a} \Rightarrow \lim_{x \to b} f(x) = a$

Bsp: $f(x) = \frac{1}{x^2}$



$\lim_{x \to 0} f(x) = \infty$

Bem: Grenzwert muss nicht existieren

$\lim_{x \to \infty} \sin(x)$ ex. nicht $\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$ ex nicht

Bsp:

- $$\lim_{x\to\infty} \frac{x^3+3}{x^4+2x+1} = \lim_{x\to\infty} \underbrace{\frac{x^3}{x^4}}_{=\frac{1}{x}\to 0} \underbrace{\frac{\left(1+\frac{3}{x^3}\right)}{\left(1+\frac{2}{x^3}+\frac{1}{x^4}\right)}}_{\to 1}$$
$$\to 0$$

- $$\cancel{\lim_{x\to\infty}} \frac{3x^2+2}{5x^2+x} = \cancel{\lim} \underbrace{\frac{x^2}{x^2}}_{=1} \underbrace{\frac{(3+2/x^2)}{(5+1/x)}}_{\to 3/5} \to 3/5 \quad (x\to\infty)$$

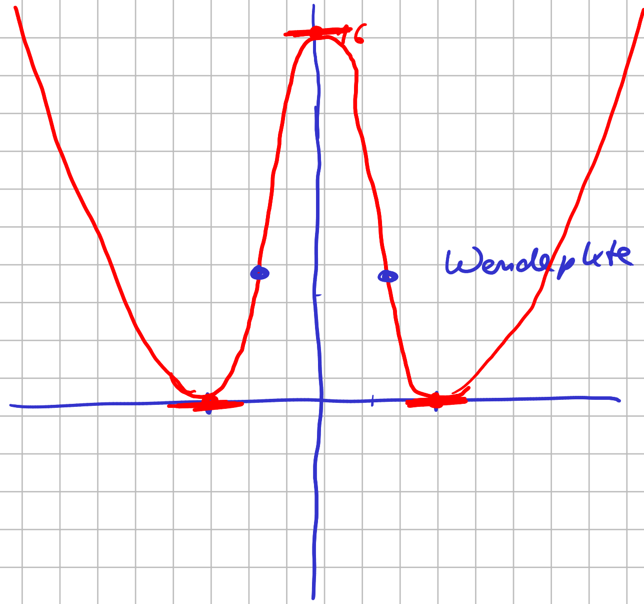
- $$\lim_{x\to 0} \frac{\sin(x)}{x} = ?$$

„$\frac{0}{0}$", „$\frac{\infty}{\infty}$" de l'Hospital

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{(\sin(x))'}{(x)'} = \lim_{x\to 0} \frac{\cos(x)}{1} = \frac{1}{1} = 1$$

Kurvendiskussion

Skizziere $f(x) = x^4 - 8x^2 + 16$



Nullstellen $f(x) = 0$

$\iff x^4 - 8x^2 + 16 \overset{!}{=} 0$

Subst. $x^2 = u$ $\quad u^2 - 8u + 16 = 0$ $\qquad u_{1,2} = \frac{8 \pm \sqrt{64-64}}{2} = \frac{8 \pm 0}{2} = 4$

$\iff x^2 = 4$

$\iff x = \pm 2$

Schnitt mit y-Achse

$f(0) = 0^4 - 8\,0^2 + 16 = 16$

Extremwerte $\quad f'(x) = 0 \iff 4 \cdot x^3 - 16x = 0$

$\iff x(4x^2 - 16) = 0$

$\iff x = 0 \vee 4x^2 = 16$

$\iff x = 0 \vee x = 2 \vee x = -2$

$f''(x) = 12x^2 - 16$

$f''(-2) = 12 \cdot 4 - 16 > 0$ $\qquad f''(0) = -16 < 0$

$f''(+2) = 12 \cdot 4 - 16 > 0$

$\Rightarrow$ lok Min bei $(-2, 0)$, $(2, 0)$

lok Max bei $(0, 16)$

Wendepunkte

$f''(x) = 0 \iff 12x^2 - 16 = 0$

$\iff x^2 = \frac{16}{12} = \frac{8}{6} = 4/3$

Verhalten für $x \to \pm\infty$

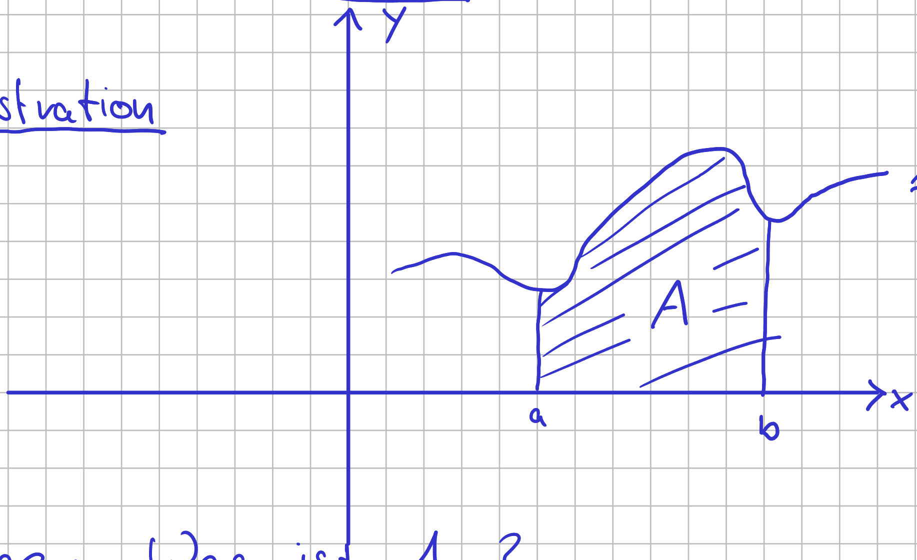
$\lim_{x \to \infty} x^4 - 8x^2 + 16 = \lim_{x \to \infty} \left(1 + \frac{8}{x^2} + \frac{16}{x^4}\right) \cdot x^4 = \infty$

$\lim_{x \to -\infty} = \infty$

Symmetrie

$f(x) = f(-x) \Rightarrow$ Achsensymmetrisch (gerade)

S3.1 Integration



Illustration

Frage: Was ist A?

$$A = \int_a^b f(x)\,dx$$

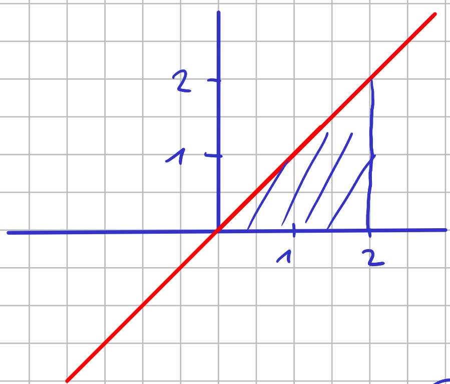
Antwort: Falls $F'(x) = f(x)$

Dann $$A = \int_a^b f(x)\,dx = F(b) - F(a)$$

(Hauptsatz Differential und Integralrechnung)

Bsp:

1



$f(x) = x$

$F(x) = \frac{1}{2}x^2$

Probe $F'(x) = 2 \cdot \frac{1}{2} x^1 = x$

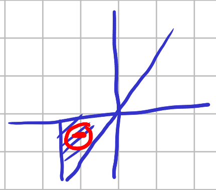
Dann

$$\int_0^2 x\,dx = F(2) - F(0) = \frac{1}{2} \cdot 2^2 - \frac{1}{2} \cdot 0^2$$

$$= \frac{1}{2} \cdot 4 = \underline{2}$$

② Fläche unter der x-Achse wird negativ gezählt

$$\int_{-2}^{0} x \, dx$$



$$= \frac{1}{2}(0)^2 - \frac{1}{2}(-2)^2 = -\frac{1}{2} \cdot 4 = \underline{\underline{-2}}$$

Schreibweisen

Falls $F'(x) = f(x)$

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

$$\int f(x)\,dx = F(x) + c$$

Bsp. $$\int_0^2 x\,dx = [\tfrac{1}{2}x^2]_0^2 = \frac{1}{2} \cdot 2^2 - \tfrac{1}{2} \cdot 0 = 2$$

$$\int x\,dx = \frac{1}{2}x^2 + c$$

wichtige Stammfunktionen

	$f(x)$	$F(x)$
$n \neq -1$	x^n	$\frac{1}{n+1} x^{n+1}$
	$\cos(x)$	$\sin(x)$
	$\sin(x)$	$-\cos(x)$
	$\frac{1}{x}$	$\ln(x)$
	e^x	e^x

Regel 1 Linearität

$$\int \alpha_1 f_1(x) + \alpha_2 f_2(x)\, dx = \alpha_1 \cdot \int f_1(x)\, dx + \alpha_2 \int f_2(x)\, dx$$

Bsp:

$$\int_{-\pi/2}^{\pi/2} 3 \cdot x^2 + \cos(x)\, dx$$

$$= 3 \cdot \int_{-\pi/2}^{\pi/2} \cdot x^2 dx + \int_{-\pi/2}^{\pi/2} \cos(x)\, dx$$

$$= 3 \cdot \left[\frac{1}{3} x^3\right]_{-\pi/2}^{\pi/2} + \left[\sin(x)\right]_{-\pi/2}^{\pi/2}$$

$$= 3 \cdot \left(\frac{1}{3}\left(\pi/2\right)^3 - 1/3\left(-\pi/2\right)^3\right) + \left(\sin(\pi/2) - \sin(-\pi/2)\right)$$

$$= \pi^3/8 + \pi^3/8 + \left(1 - (-1)\right)$$

$$= 2 \cdot \pi^3/8 + 2 = \underline{\underline{\pi^3/4 + 2}}$$

Regel 2 partielle Integration

$$G' = g$$

$$\int \underset{\downarrow}{f} \underset{\uparrow}{g} = fG - \int f'G$$

Bsp:

$$\int \underset{\downarrow}{x} \cdot \underset{\uparrow}{\cos(x)}\, dx = x \cdot \sin(x) - \int \sin(x)\, dx$$

$$= x \cdot \sin(x) - \left(-\cos(x)\right) + C$$

$$= x \cdot \sin(x) + \cos(x) + C$$

Probe: $\left(x\cdot\sin(x)+\cos(x)+c\right)'$

$$= \sin(x) + x\cdot\cos(x) - \sin(x) = x\cdot\cos(x)$$

(2.) $$\int_0^1 \underset{\downarrow}{x}\cdot \underset{\uparrow}{e^x} = \left[x\cdot e^x\right]_0^1 - \int_0^1 e^x = 1\cdot e^1 - 0\cdot e^0 - \left[e^x\right]_0^1$$

$$= e^1 - \left(e^1 - e^0\right) = e^0 = \underline{\underline{1}}$$

Regel 3 Substitution

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(y)\,dy$$

Bsp:

(1) $$\int_{\sqrt{\pi}}^{\sqrt{2\pi}} x\cdot\sin\left(x^2\right)dx \underset{\substack{u=x^2\\ \frac{du}{dx}=2x\\ \Leftrightarrow\ dx=\frac{1}{2x}du}}{=} \int_{(\sqrt{\pi})^2}^{(\sqrt{2\pi})^2} \cancel{x}\cdot\sin(u)\cdot\frac{1}{2\cancel{x}}\,du$$

$$= \frac{1}{2}\int_{\pi}^{2\pi}\sin(u)\,du$$

$$= \frac{1}{2}\cdot\left[-\cos(u)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2}\cdot\left(-\cos(2\pi) - \left(-\cos(\pi)\right)\right)$$

$$= \frac{1}{2}\left(-\cos(2\pi) + \cos(\pi)\right)$$

$$= \frac{1}{2}\left(-1 + (-1)\right)$$

$$= -1$$

② Unbestimmtes Integral

$$\int e^{2x+3}\,dx \underset{\substack{u=2x+3\\ \frac{du}{dx}=2\\ \Rightarrow dx=\frac{1}{2}du}}{=} \int e^{u}\cdot\frac{1}{2}\,du \;\Big|_{u=2x+3}$$

$$= \frac{1}{2}e^{u}+C \;\Big|_{u=2x+3}$$

$$= \frac{1}{2}\cdot e^{2x+3}+C$$

<u>Probe</u>: $\left(\frac{1}{2}\cdot e^{2x+3}\right)' = \frac{1}{2}\cdot e^{2x+3}\cdot(2) = e^{2x+3}$

S3.3 Komplexe Zahle

Warum? $x^2+1=0$ hat keine Lsg. in $\mathbb{R}$

Deswegen $\mathbb{C} = \{a + i\cdot b \mid a, b \in \mathbb{R}\}$, $\mathbb{R} \subseteq \mathbb{C}$

„Komplexe Zahlen"

mit $i^2 = i\cdot i = -1$

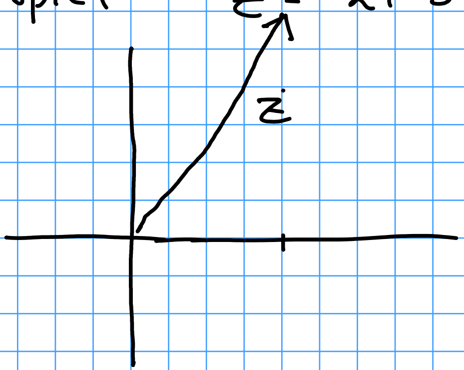
Für eine komplexe Zahl $z = a + ib$

ist $\mathrm{Re}(z) = a$ Realteil von z

$\mathrm{Im}(z) = b$ Imäginarteil von z

Beispiel $z = 2 + 3i$ hat Realteil 2

Imag. teil 3



Gaußsche Zahlenebene

Rechnen in $\mathbb{C}$

genauso wie in $\mathbb{R}$ mit $i^2 = -1$

Also z.B.

① Addition $(2+3i) + (1+5i) = (2+1) + (3+5)\cdot i$

$= 3 + 8i$

② Multiplikation $(1+2i)\cdot(2+3i)$

$= 1\cdot 2 + 1\cdot 3i + 2i\cdot 2 + (2i)\cdot(3i) = 2 + 3i + 4i + 6\cdot\underbrace{i^2}_{-1}$

$= (2-6) + i(3+4)$

$= -4 + 7i$

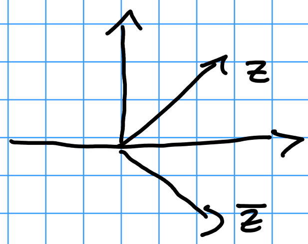
Bsp: $(-1+2i)\cdot i + 3i - 2 = -1\cdot i + (2i)\cdot i + 3i - 2$

$= -i - 2 + 3i - 2$

$= \underline{\underline{2i - 4}}$

③ Neu: $z = a + ib$, $\overline{z} = a - ib$ Komplex

z.B $\overline{3+2i} = 3 - 2i$



④ Betrag $z = a+ib$

$|z| = \sqrt{a^2+b^2}$ = Länge des Pfeils

auch $|z|^2 = z \cdot \overline{z}$

<u>Bsp:</u> ① $|1+i| = \sqrt{1+1} = \sqrt{2}$

$|i| = \sqrt{0^2+1^2} = 1$

⑤ Division

$$\frac{3+4i}{1+2i} = \frac{3+4i}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{3-6i+4i+(4i)\cdot(-2i)}{1-\cancel{2i}+\cancel{2i}+(2i)(-2i)}$$

$$= \frac{3-2i-8i^2}{1-4\cdot i^2} = \frac{3+8-2i}{1+4} = \frac{11-2i}{5}$$

$$= \frac{11}{5} - \frac{2}{5}i$$

<u>Bsp:</u>

$$(x-(1+2i))(x-(1-2i))$$

$$= x^2 - (1+2i)\cdot x - (1-2i)x + (1+2i)(1-2i)$$

$$= x^2 - 2x + 5$$

$$x_{1,2} = \frac{2 \pm \sqrt{4-20}}{2}$$

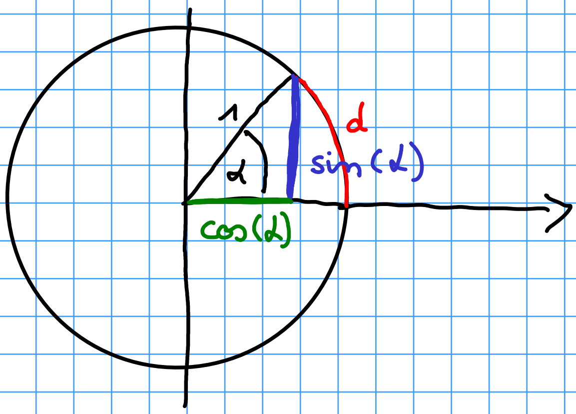
$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm \sqrt{-1} \cdot \sqrt{16}}{2}$$

$$= \frac{2 \pm i \cdot 4}{2} = 1+2i,\ 1-2i$$

Polar Koordinaten

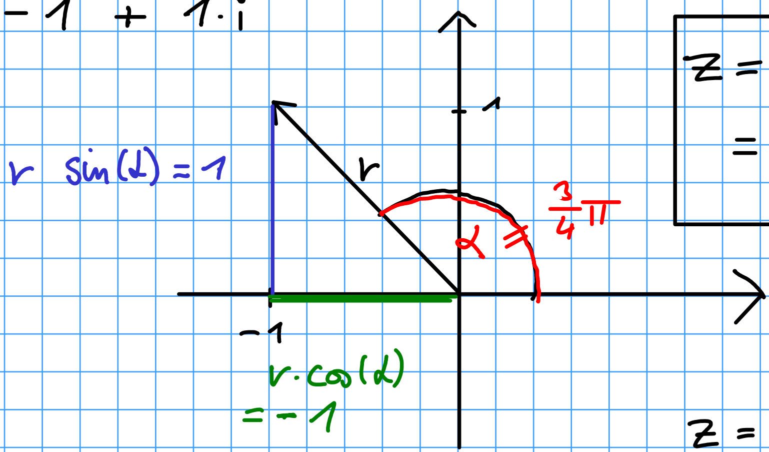
Einschub: Winkel in Bogenmaß



α °	α rad
90°	$\pi/2$
180°	π
360°	2π
x°	$\frac{x}{360}\cdot 2\pi$

Polar Koordinaten

Bsp: $z = -1 + 1 \cdot i$



$$\boxed{\begin{aligned} z &= r\cdot\cos(\alpha) + i\cdot r\cdot\sin(\alpha) \\ &= r\cdot e^{i\alpha}\end{aligned}}$$

Hier

$$z = \cos\left(\tfrac{3}{4}\pi\right) + i\cdot\sin\left(\tfrac{3}{4}\pi\right)$$

$$= e^{3/4\pi\cdot i}$$

Euler Formel

$$\boxed{e^{\pi i} + 1 = 0}$$